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Product Properties of Hilbert Transforms

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Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ with $1 < p < \infty$, $1 < q < \infty$ and let Hf, Hg be their respective Hilbert transforms. We give a simple proof of the identity $Hf \cdot Hg - f \cdot g = H(f \cdot Hg + g \cdot Hf)$ a.e. and of its inverse in the case $(1/p) + (1/q) \leqslant 1$ which includes the cases already considered by Cossar and Tricomi. We next derive applications, especially to boundary values of analytic functions.

1. NOTATION

We consider complex-valued functions on $\mathbb{R}$ and use the following notation:

$\mathfrak{B}(E, F)$: set of bounded linear operators from vector space E to vector space F.

C_0: set of continuous functions vanishing at infinity.

$\mathfrak{F}$: Fourier operator defined (i) for $f \in L^1$, by $(\mathfrak{F}f)(x) = \int_{-\infty}^{+\infty} f(t)\, e^{-ixt}\, dt$, (ii) for $f \in L^2$, by limits in L^2 of analogous truncated integrals.

$\mathfrak{F}^*$: inverse Fourier operator defined by

$$(\mathfrak{F}^* f)(x) = \lim_{y \to 0_+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t)\, e^{itx} e^{-|t|y}\, dt$$

a.e. on the space of functions f such that this limit exists a.e.

Hf: Hilbert transform of f, defined a.e. by

$$(Hf)(x) = \lim_{N \to \infty} \frac{1}{\pi} \int_{1/N}^{N} \frac{f(t-u) - f(t+u)}{u}\, du$$

$$\text{for} \quad f \in L^p, \quad 1 \leqslant p < \infty.$$

σ: multiplication operator by function $\sigma(x) = -i \operatorname{sign} x$.

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2. Product Properties

2.1. Preliminary Lemma. *If $F \in L^2$, $G \in L^2$, then the following identity holds in C_0:*

$$(\sigma F * \sigma G) - (F * G) = \sigma[(G * \sigma F) + (\sigma G * F)]. \tag{1}$$

Proof. It suffices to point out that, as a straightforward calculation shows, both sides of (1) are equal to

$$-2 \operatorname{sign} x \int_0^x F(t)\, G(x - t)\, dt.$$

2.2. Theorem. *If $f \in L^p$, $g \in L^q$ with $1 < p < \infty$, $1 < q < \infty$ and $1/r = (1/p) + (1/q) \leqslant 1$, then*

$$Hf \cdot Hg - f \cdot g = H(f \cdot Hg + g \cdot Hf) \qquad \text{a.e.}, \tag{2}$$

$$f \cdot Hg + g \cdot Hf = -H(Hf \cdot Hg - f \cdot g) \qquad \text{a.e.} \tag{3}$$

When $r = 1$, this implies in particular that $H(f \cdot Hg + g \cdot Hf) \in L^1$ and $H(Hf \cdot Hg - fg) \in L^1$.

Proof. (a) We first consider the case $p = q = 2$. By Lemma 2.1 applied with $F = \mathfrak{F}f$, $G = \mathfrak{F}g$, and since $\mathfrak{F}Hf = \sigma\mathfrak{F}f$, $\mathfrak{F}Hg = \sigma\mathfrak{F}g$ in L^2, we have

$$(\mathfrak{F}Hf * \mathfrak{F}Hg) - (\mathfrak{F}f * \mathfrak{F}g) = \sigma[(\mathfrak{F}g * \mathfrak{F}Hf) + (\mathfrak{F}f * \mathfrak{F}Hg)]$$

in C_0. This may be written

$$\mathfrak{F}(Hf \cdot Hg - f \cdot g) = \sigma\mathfrak{F}(g \cdot Hf + f \cdot Hg), \tag{4}$$

for

$$\mathfrak{F}f * \mathfrak{F}g = \mathfrak{F}(fg) \qquad \text{whenever} \qquad f \in L^2,\ g \in L^2.$$

We then obtain identity (2) by making $\mathfrak{F}^*$ operate on both sides of (4) and by using Proposition 8.3.4 of [1].

(b) Let us now suppose $1/r = (1/p) + (1/q) \leqslant 1$. Then, there exist $f_n \in L^2 \cap L^p$, $g_n \in L^2 \cap L^q$ such that $f_n \to f$ in L^p norm, $g_n \to g$ in L^q norm and

$$(Hf_n)(Hg_n) - f_n g_n = H(f_n Hg_n + g_n Hf_n) \qquad \text{a.e.} \tag{5}$$

for every n.

When $r = 1$, the first member of the latter equality converges to $Hf \cdot Hg - f \cdot g$ in L^1 norm, while the second one converges to $H(f \cdot Hg + g \cdot Hf)$ in measure ([3], III, Theorem 6). Consequently, one can find a subsequence $H(f_{n_k} \cdot Hg_{n_k} + g_{n_k} \cdot Hf_{n_k})$ converging a.e. both to $Hf \cdot Hg - f \cdot g$ and

$H(f \cdot Hg + gHf)$, which implies identity (2) and in particular that $H(f \cdot Hg + g \cdot Hf) \in L^1$. Identity (3) then follows from Proposition 8.2.10 of [1].

When $r > 1$, both members of (5) converge in L^r norm since $H \in \mathfrak{B}(L^r, L^r)$ for $r > 1$. We thus obtain identity (2) a.e. Identity (3) follows by the well-known property $H^2 f = -f$ in L^r for $r > 1$.

Remark. The preceding theorem extends Theorem IV of [5] and, a fortiori, Lemma 16 of [2].

2.3. COROLLARIES. *Let f and g be as in the preceding theorem. Then.*

(1)

$$P \cdot V \cdot \int_{-\infty}^{+\infty} \frac{(Hf)(x) - (Hf)(t)}{x - t} g(t)\, dt$$

$$= -P \cdot V \cdot \int_{-\infty}^{+\infty} \frac{f(x) - f(t)}{x - t} (Hg)(t)\, dt \qquad \text{a.e.}$$

(2) *$Hf = if$ (resp. $Hf = -if$) and $Hg = ig$ (resp. $Hg = -ig$) imply $H(fg) = ifg$ (resp. $H(fg) = -ifg$). In particular, $H(uv) = iuv$ for $u = f - iHf$, $v = g - iHg$; similarly, $H(uv) = -iuv$ for $u = f + iHf$, $v = g + iHg$.*

3. CONNECTION WITH BOUNDARY VALUES OF ANALYTIC FUNCTIONS

For $f \in L^p$, $1 < p < \infty$, let us define Cf by

$$(Cf)(z) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{t - z}\, dt$$

for $z \in \mathbb{C} \setminus \mathbb{R}$.

One may easily deduce from [3, p. 67]. that

$$(C^+ f)(x) = \lim_{y \to 0_+} (Cf)(x + iy)$$

and

$$(C^- f)(x) = \lim_{y \to 0_-} (Cf)(x + iy)$$

exist a.e. and satisfy

$$C^+ f = (1/2i)(-Hf + if), \qquad C^- f = (-1/2i)(Hf + if). \tag{6}$$

We have the following corollaries concerning C^+ and C^-:

COROLLARY 3. *If* $f \in L^p$, $g \in L^q$ *with* $1 < p < \infty$, $1 < q < \infty$ *and* $1/r = (1/p) + (1/q) \leqslant 1$, *then* $H(C^+f \cdot C^+g) = -i(C^+f \cdot C^+g)$ *and* $H(C^-f \cdot C^-g) = +i(C^-f \cdot C^-g)$ a.e. *In particular,* $H(C^+f \cdot C^+g) \in L^1$ *and* $H(C^-f \cdot C^-g) \in L^1$ *when* $r = 1$.

COROLLARY 4. *If f and g are defined as in the preceding corollary, then*

$$C^+(C^+f \cdot C^+g) = C^+f \cdot C^+g, \qquad C^-(C^+f \cdot C^+g) = 0 \qquad \text{a.e.}$$

In particular, $C^+(C^+f \cdot C^+g) \in L^1$ *when* $r = 1$. *Analogous conclusions hold for* C^-:

$$C^-(C^-f \cdot C^-g) = -C^-f \cdot C^-g, \qquad C^+(C^-f \cdot C^-g) = 0 \qquad \text{a.e.}$$

Proofs. It is easy to deduce Corollary 3 from relations (6) and Corollary 2. It is the same for the assertions of Corollary 4 in the case $r > 1$. When $r = 1$, it is not even evident that $C^+(C^+f \cdot C^+g)$ exists a.e. and we must proceed otherwise. In this purpose, we notice that, for $F \in L^1$, we may write

$$\begin{aligned}(CF)(x + iy) &= \frac{1}{2\pi} \int_0^{\infty} (\mathfrak{F}F)(t)\, e^{itx} e^{-ty}\, dt \qquad (y > 0),\\ &= -\frac{1}{2\pi} \int_{-\infty}^{0} (\mathfrak{F}F)(t)\, e^{itx} e^{-ty}\, dt \qquad (y < 0),\end{aligned}$$

since $(2i\pi)(CF)(z) = \int_{-\infty}^{+\infty} F(t)(\mathfrak{F}g_z)(t)\, dt$, where $z = x + iy$, $g_z(t) = ie^{itz}U(t)$ for $y > 0$, $g_z(t) = -ie^{itz}U(-t)$ for $y < 0$, $U(t) = 0$ for $t < 0$, $U(t) = 1$ for $t > 0$. Moreover, when $F = C^+f \cdot C^+g$ and $r = 1$, we have $HF = -iF$ by Corollary 3. This implies $\mathfrak{F}HF = -i\mathfrak{F}F$ in C_0 and, consequently, $\sigma\mathfrak{F}F = -i\mathfrak{F}F$ by Proposition 8.3.1 of [1]. Thus, $(\mathfrak{F}F)(x) = 0$ for every $x \leqslant 0$, and

$$\begin{aligned}(C^+F)(x) &= \lim_{y \to 0_+} \int_{-\infty}^{+\infty} (\mathfrak{F}F)(t)\, e^{itx} e^{-|t|y}\, dt = F(x) \qquad \text{a.e.},\\ (C^-F)(x) &= 0,\end{aligned}$$

which implies the first required identities.

Properties concerning C^- are deduced by similar arguments.

Remark 1. From relations (6) and the above corollary, one can derive the following identities too, under the hypotheses of the preceding corollaries:

$$\begin{aligned}C^+f \cdot C^+g &= C^+(f \cdot C^+g + g \cdot C^+f - fg) \qquad \text{a.e.},\\ C^-f \cdot C^-g &= C^-(f \cdot C^-g + g \cdot C^-f + fg) \qquad \text{a.e.}\end{aligned}$$

Remark 2. When $r = 1$, an alternative proof of the first property of C^+ in Corollary 4 is the following one.

One has $C^+f \cdot C^+g = h + iHh$ with $h = (i/4)(fHg + gHf)$. By Theorem 2.2, h as well as Hh are in L^1; consequently, $F = C^+f \cdot C^+g \in L^1$ and $HF \in L^1$. Moreover, $(CF)(z) = (1/2i)[(-Hn_y) * F - i(n_y * F)](x)$ for $y > 0$, where $z = x + iy$ and $n_y(x) = (1/\pi)(y/(x^2 + y^2))$ is the Poisson kernel. By Proposition 8.2.3 of [1], this implies $(CF)(z) = (1/2i)[n_y * (-HF + iF)](x)$ and consequently, $C^+F = F$, which is the required identity.

The existence a.e. and in the L^1 norm of C^+F could be derived from [4, pp. 220, 221] too.

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